$$|\pi z(t;\ \varepsilon)| \geqslant {}^1/_2 \min\ \{r_*,\ |\pi z_*|\} > 0, \quad t \in I_0$$

Setting $\mu_*(s) = \max\ \{\rho_*(s),\ p_*(s)\}$, from (8.12), (8.13) and (8.16) we have

$$|\pi z(t;\ \varepsilon)| \geqslant \mu_*(t - \beta_n), \quad t \in I_n, \quad n = 1, 2, \ldots \tag{8.18}$$

Let us show that $\mu_*(s) > 0$ on $[0,\ \theta]$. In fact (see (8.17)), $p_*(s) \geqslant r_* / 2 > 0$, $\tau_* \leqslant s \leqslant \theta$. By virtue of the definition of τ_* (see (8.4)) we have

$$\rho_*(s) \geqslant r_0 - Dc(\tau_*) - \varepsilon_* c(1 + e^{K\theta}) \geqslant r_0 / 2 - \varepsilon_*\ (e^{2K\theta} - 1) / K$$

on the interval $[0,\ \tau_*]$. Hence, according to the definition of ε_* we have $\rho_*(s) \geqslant r_0 / 4 > 0,\ s \in [0,\ \tau_*]$. The positiveness of $\mu_*(s)$ is proved.

Since $\mu_*(s)$ is continuous, we have that $l_* = \min\limits_{s \in [0,\ \theta]} \mu_*(s) > 0$, so that formulas (8.16) and (8.18) guarantee l-escape in problem (1.1) for $\varepsilon \in [0,\ \varepsilon_*]$ and $z(t_*;\ \varepsilon) = z_*$.

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NONCANONICAL INVARIANTS OF HAMILTONIAN SYSTEMS

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We explain the character of simplifications which can be carried out in the Hamiltonian function of a nonresonant system using the formal, noncanonical transformations. We show the symmetries of such systems, which are not generated by their first integrals. Using a Hamiltonian system with two degrees of freedom we show that the noncanonical transformations retaining its normal form but with

displaced coefficients, exist also when resonances are present. Formulas defining these transformations are given.

1. Statement of the problem and the result. The coefficients of a Hamiltonian in its normal form are canonical invariants [1] (i. e. remain unchanged under any canonical transformations which preserve the normal form). Although the group of canonical transformations is infinite dimensional, it is in certain sense narrow, since it is strictly required to transform any Hamiltonian system into another Hamiltonian system. At the same time, the maximal group of transformations preserving only a certain subclass of Hamiltonian systems will be, generally speaking, no longer a canonical one. On the other hand, its action on the chosen subclass will become more effective since the group is more general than the canonical group. The appearance of the symmetries in the Hamiltonian systems not generated by the first integrals can be explained by the analogous widening of the groups on the subclasses. We can use the normal form as the Hamiltonian subclass under investigation. In this case we can explain the character of simplifications for the nonresonant systems which can be carried out on the Hamiltonians, using formal, noncanonical transformations. In other words, the following theorem holds.

Theorem. Let the conditions

$$\det\ (\beta_{ij}) \neq 0, \quad \beta_{ij} \neq 0, \quad i,\ j \leqslant n \tag{1.1}$$

hold for the Hamiltonian

$$H = \sum_i \alpha_i u_i + \sum_{i,j} \beta_{ij} u_i u_j + H_5 + H_6 + \ldots, \quad u_i = x_i^2 + p_i^2$$

of a stationary real nonresonant system. Then using a formal noncanonical change of variables, we can transform the function H to the form

$$H = \sum_i \hat{\alpha}_i u_i + \sum_{i,j} \beta_{ij} u_i u_j + \sum_{k=1}^{n} f_k(u_k) \tag{1.2}$$

in which none of the formal series $f_k(u_k) = a_{k3} u_k^3 + a_{k4} u_k^4 + \ldots$ can any longer be altered.

The operators corresponding to the one-parameter symmetry groups have the form

$$\sum_{k=1}^{n} \Phi_k(u_1, \ldots, u_n) \left(p_k \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial p_k} \right)$$

where Φ_k are arbitrary functions, and the above transformation will be canonical if $\Phi_k = \partial\varphi / \partial u_k$.

Let us now consider a Hamiltonian system with two degrees of freedom in the presence of a resonance of order $q = m_1 + m_2$. By the Moser theorem [1] the Hamiltonian of the system can be reduced to the normal form which is given in the canonical polar coordinates x_ν, p_ν by

$$H = \sum_{k=0}^{\infty} (\rho_1^{m_1} \rho_2^{m_2})^{k/2} f_{(k)}(\rho_1, \rho_2) (a_k e^{ik\theta} + \bar{a}_k e^{-ik\theta})$$

where $f_k(\rho)$ are formal power series in integral powers of ρ ; $\theta = m_1\varphi_1 + m_2\varphi_2$ is the resonance phase, and $f_{(0)}(\rho) = m_2\rho_1 - m_1\rho_2 + O(\rho^2)$, $\alpha_1 = m_2$, $\alpha_2 = -m_1$

$$(x_\nu = \sqrt{\rho_\nu} \cos \varphi_\nu, \quad p_\nu = \sqrt{\rho_\nu} \sin \varphi_\nu)$$

Using the new canonical variables

$$u_1 = \frac{1}{q}(m_2\rho_1 - m_1\rho_2), \qquad u_2 = \frac{1}{q}(\rho_1 + \rho_2)$$
$$\theta = m_1\varphi_1 + m_2\varphi_2, \qquad \alpha = \varphi_1 - \varphi_2$$

we can write the equations of motion in the form

$$u_1^{\cdot} = 0, \quad \alpha^{\cdot} = -\frac{\partial H}{\partial u_1}, \quad u_2^{\cdot} = \frac{\partial H}{\partial \theta}, \quad \theta^{\cdot} = -\frac{\partial H}{\partial u_2}$$

Let us denote by H^* the result of replacing the coefficients a_k in the H by their infinitesimal displacements ζ_k. Then H^* is given by the formula

$$H^* = f(u_1, \ H) - \left(\varphi_1(u_1)\frac{\partial H}{\partial u_1} + \varphi_2^\circ\frac{\partial H}{\partial u_2} + \psi^\circ\frac{\partial H}{\partial \theta}\right) \tag{1.3}$$

where the functions f, φ_2° and ψ° satisfy the following unique equation:

$$\frac{\partial H}{\partial u_1}\frac{\partial \varphi_2^\circ}{\partial u_2} - \frac{\partial H}{\partial u_2}\frac{\partial \varphi_2^\circ}{\partial u_1} + \frac{\partial H}{\partial u_1}\frac{\partial \psi^\circ}{\partial \theta} - \frac{\partial H}{\partial \theta}\frac{\partial \psi^\circ}{\partial u_1} = \Phi^*(u_1, \ H)\frac{\partial H}{\partial u_1} - \frac{\partial f}{\partial u_1} \tag{1.4}$$

and $\varphi_1(u_1)$ and $\Phi^*(u_1, \ H)$ are arbitrary functions of their arguments.

Certain complications arise when the formulas (1.3) and (1.4) are used to simplify the Hamiltonian. We shall just say that H^*, as can be seen from the formulas, can vary over wide limits, and this makes possible the removal of a large number of terms from the expansion of the Hamiltonian H.

Detailed derivation of the formulas (1.3) and (1.4) is not given here.

2. Proof of the theorem. It is expedient to pass, in the real Hamiltonian system, to the complex variables $z_k = x_k + ip_k$, $\bar{z}_k = x_k - ip_k$. If $H \to -2iH$, then the change is canonical and the system can now be written in the form

$$z_k^{\cdot} = \partial H / \partial \bar{z}_k, \quad \bar{z}_k^{\cdot} = -\partial H / \partial z_k$$

Using the Birkhoff transformation we reduce the system to the normal form [2], so that $H = H(u_1, \ldots, u_n)$, $u_k = z_k\bar{z}_k$ and

$$z_k^{\cdot} = \frac{\partial H}{\partial u_k} z_k, \quad \bar{z}_k^{\cdot} = -\frac{\partial H}{\partial u_k}\bar{z}_k$$

The displacement operator along the trajectories assumes the form

$$L = \sum_{k=1}^{n} \frac{\partial H}{\partial u_k}\left(z_k\frac{\partial}{\partial z_k} - \bar{z}_k\frac{\partial}{\partial \bar{z}_k}\right)$$

Let

$$Z = \sum_{j=1}^{n}\left(\xi_j(u)\, z_j\frac{\partial}{\partial z_j} + \bar{\xi}_j(u)\, \bar{z}_j\frac{\partial}{\partial \bar{z}_j}\right) + \sum_i \zeta_i(a)\frac{\partial}{\partial a_i} \equiv$$
$$\equiv Y + \sum_i \zeta_i(a)\frac{\partial}{\partial a_i}$$

be an operator (of an infinitesimal transformation) corresponding to the one-parameter group G of transformations of the space $\{z, \ \bar{z}, \ a\}$ into itself (a_i are the coefficients of expansion in powers of u, of the Hamiltonian H). The necessary condition for the transformations belonging to G to transform a Hamiltonian system into another Hamiltonian system (and consequently every motion of the initial system into a motion of the transformed system) is, that the operators L and Z commute, i.e. $[L, Z] = 0$. From this we have

$$[L, Y] = \sum_{k=1}^{n} \frac{\partial H^*}{\partial u_k}\left(z_i \frac{\partial}{\partial z_i} - \bar{z}\frac{\partial}{\partial \bar{z}_i}\right) \qquad (2.1)$$

where

$$H^* = \sum_i \zeta_i(a) \frac{\partial H}{\partial a_i} \qquad (2.2)$$

Since H depends on the parameters a_i linearly, the function H^* represents the result of replacing the coefficients a_i in the Hamiltonian H by their infinitesimal displacements $\zeta_i(a)$.

Let us find the commutator $[L, Y]$. Omitting the detailed computations, we find

$$[L, Y] = -\sum_{k=1}^{n}\left(\sum_{j=1}^{n} u_j(\xi_j + \bar{\xi}_j)\frac{\partial^2 H}{\partial u_k \partial u_j}\right)\left(z_k \frac{\partial}{\partial z_k} - \bar{z}\frac{\partial}{\partial \bar{z}_k}\right)$$

Comparison with (2.1) yields

$$\frac{\partial H^*}{\partial u_k} = -\sum_{j=1}^{n} u_j \psi_j(u) \frac{\partial^2 H}{\partial u_k \partial u_j}, \quad \psi_j(u) = \xi_j + \bar{\xi}_j \quad (k \leqslant n) \qquad (2.3)$$

The tranformations of the phase variables z_k, $\bar{z}_k$ under which all coefficients a_i of H remain unchanged, yield the symmetry group of the initial system. For this reason the symmetry group is generated by those functions ξ_j for which $H^* = 0$. From (2.3) we see that if $\det(\partial^2 H/\partial u_k \partial u_j) \neq 0$ (i.e. the conditions (1.1) are satisfied) then $\psi_j(u) = 0$, $j \leqslant n$ and $\xi_j = i\Phi_j(u_1, \ldots, u_n)$ where Φ_j are arbitrary functions. If conditions (1.1) are not satisfied, then the symmetry group will contain $m = n - \text{rank}(\partial^2 H / \partial u_k \partial u_j)$ additional independent generatrices.

Let us now consider the transformations which displace the coefficients of the Hamiltonian H.

If $\psi_j(u)$ can be chosen so that H^* contains a term with the coefficient equal to unity, then a term of the same designation can be annihilated in H by means of a formal transformation. This transformation leaves unaffected all the coefficients in H accompanying the terms on which H^* does not depend. The remaining coefficients of the expansion of H will be transformed in some manner. If H^* could be chosen arbitrarily, then all terms in H could be annihilated. This however cannot be done, irrespective of the fact that by virtue of the condition (1.1) all functions $u_1\psi_1, \ldots, u_n\psi_n$ can be found from the equations (2.3) in the form of formal power series for any H^* given in the form of a series (or a polynomial). The fact is, that the series computed for $u_j\psi_j$ must be divisible by u_j. This condition can always be fulfilled by setting $H^* = u_1^{m_1} \ldots u_n^{m_n}$ for $m_1 \geqslant 2, \ldots, m_n \geqslant 2$. It is evident that in this case the power series for $\psi_j(u)$ exists, consequently all terms of the type shown above can be annihilated in H.

It can easily be shown that a series for H^* in which the functions $\psi_j(u)$ are obtained by the power series in positive powers of u can be given, for the case of $n = 2$ (the general case can be dealt with in a similar manner), in the form

$$H^* = \int \frac{\partial H^*}{\partial u_1}\bigg|_{u_2=0} du_1 + \int \frac{\partial H^*}{\partial u_2}\bigg|_{u_1=0} du_2 + u_1 \frac{\partial H^*}{\partial u_1}\bigg|_{u_1=0} + u_2 \frac{\partial H^*}{\partial u_2}\bigg|_{u_2=0} =$$

$$-\int \omega_1 \frac{\partial^2 H}{\partial u_1^2}\bigg|_{u_2=0} du_1 - \int \omega_2 \frac{\partial^2 H}{\partial u_2^2}\bigg|_{u_1=0} du_2 - u_1\omega_2 \frac{\partial^2 H}{\partial u_1 \partial u_2}\bigg|_{u_1=0} - u_2\omega_1 \frac{\partial^2 H}{\partial u_1 \partial u_2}\bigg|_{u_2=0}$$

$$(\omega_1 = u_1\psi_1(u_1, 0), \quad \omega_2 = u_2\psi_2(0, u_2))$$

It is clear from the above expression that by virtue of the condition $\beta_{12} \neq 0$, the terms appearing in the binomials $a_1 u_1^m + b u_1^{m-1} u_2$ and $c u_1 u_2^{m-1} + d u_2^m$ are transformed simultaneously for each order. For this reason only one term can be annihilated in each binomial. In particular, setting consecutively $\omega_1 = u_1^m$ and $\omega_2 = u_2^m$, we arrive at the expression (1.2).

3. A scheme for proving the formulas (1.3) and (1.4). Let us denote by L the displacement operator along the trajectories written in the variables u_1, u_2, θ and α, and by X the operator of transformation of the following Hamiltonian system:

$$L = \frac{\partial H}{\partial \theta}\frac{\partial}{\partial u_2} - \frac{\partial H}{\partial u_2}\frac{\partial}{\partial \theta} + \frac{\partial H}{\partial u_1}\frac{\partial}{\partial \alpha}$$

$$X^* = \xi_1 \frac{\partial}{\partial u_1} + \xi_2 \frac{\partial}{\partial u_2} + \psi\frac{\partial}{\partial \theta} + \xi\frac{\partial}{\partial \alpha} + \sum_j \zeta_j(a)\frac{\partial}{\partial a_j} \equiv$$

$$X + \sum_j \zeta_j(a)\frac{\partial}{\partial a_j}$$

Transforming the equations obtained from the condition of invariance of $[L, X^*] = 0$, we obtain $\xi_1 = \xi_1(u_1, H), H^* = -XH + f(u_1, H)$ and

$$\frac{\partial \xi_1}{\partial H}\frac{\partial H}{\partial u_1} + \frac{\partial \xi_2}{\partial u_2} + \frac{\partial \psi}{\partial \theta} + \frac{\partial \xi}{\partial \alpha} = \Phi(u_1, H) \tag{3.1}$$

where ξ_1, f and Φ are arbitrary functions of u_1 and H, and H^* is determined exactly as in (2.2). Further manipulations yield

$$\begin{aligned} \xi_2 &= \varphi_2^\circ + \left[\alpha\left(\frac{\partial f}{\partial H} - \Phi\right) + \xi\right]\frac{\partial H}{\partial \theta} \bigg/ \frac{\partial H}{\partial u_1} \\ \psi &= \psi^\circ - \left[\alpha\left(\frac{\partial f}{\partial H} - \Phi\right) + \xi\right]\frac{\partial H}{\partial u_2} \bigg/ \frac{\partial H}{\partial u_1} \end{aligned} \tag{3.2}$$

Here φ_2° and ψ° are arbitrary functions independent of α. From (3.2) follows (1.3). Substituting the expressions (3.2) into the equation of motion containing $\partial H^* / \partial u_1$ yields the formula (1.4) in which $\Phi^* = \Phi - \partial f / \partial H$. The functions φ° and ψ° are given by (1.4), while ξ_2 and ψ by (3.2). After this the function $\xi^* \equiv [\xi + \alpha(\partial f / \partial H - \Phi)]\, \partial H / \partial u_1$ can be found from (3.1) in the form of a formal series in α. Finally, we obtain

$$\xi = \xi^* \frac{\partial H}{\partial u_1} + \alpha\left(\Phi - \frac{\partial f}{\partial H}\right), \quad \xi_2 = \varphi_2^\circ + \xi^*\frac{\partial H}{\partial \theta}, \quad \psi = \psi^\circ - \xi^*\frac{\partial H}{\partial u_2}$$

4. Note. The questions of convergence were not considered here. The general problem of convergence was studied in [3]. The analyticity of the normalizing transformation for the second order systems which are Hamiltonian in the linear approximation, was proved by the author (*). (A complete proof of this obtained by the author

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*) Markhashov, L. M., On the analytic equivalence of the systems of ordinary differential equations with resonances. Preprint №36 of the Inst. of the Problems of Mechanics, Akad. Nauk SSSR, Moscow, 1974.

does not, in fact, contain any of the omissions noted in the text of [4]). The presence of a finite number of formal invariants of the second order non-Hamiltonian systems with resonances was established earlier [5]. The same aspect was studied for the multi-dimensional systems by the author in [6] and (simultaneously and independently) in [7].

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EVENTUAL STABILITY OF DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE

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For differential systems of neutral type we examine one of the formulations of the finite-time interval stability problem, i.e., technical stability. By the Liapunov-Krasovskii method [1 — 3] we obtain sufficient conditions for technical stability and for the so-called contracting technical stability. Similar investigations for ordinary differential equations were carried out in [4] and for equations with a lagging argument, in [5, 6].

1. We are given a system of differential equations

$$\frac{d}{dt} D(x_t(\theta), t) = f(x_t(\theta), t), \quad D(x_t(\theta), t) \equiv x(t) - g(x_t(\theta), t) \tag{1.1}$$

$$g(x(\theta), t) \equiv \int_{-\tau}^{0} [d_\theta \mu(\theta, t)]\, x(\theta)$$

Here the vector function $x_t(\theta) \equiv x(t+\theta)$ belongs for all $t \geqslant 0$ to the space $C_0 \equiv C([-\tau, 0], R^n)$ with the norm $\| x(\theta) \| = \sup (\mid x_i(\theta) \mid$ for $-\tau \leqslant \theta \leqslant 0$, $i = 1, 2, \ldots, n)$; $\mu(\theta, t)$ is an $(n \times n)$-matrix of functions continuous in $t \in [0, \infty)$ and of bounded variation in θ, for which a continuous function $l_0(s)$, nonde-